Hereditary Efficiently Dominatable Graphs*

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Abstract: An efficient dominating set (or perfect code) in a graph is a set of vertices the closed neighborhoods of which partition the graph's vertex set. We introduce graphs that are hereditary efficiently dominatable in that sense that every induced subgraph of the graph contains an efficient dominating set. We prove a decomposition theorem for (bull, fork, C_4)-free graphs, based on which we characterize, in terms of forbidden induced subgraphs, the class of hereditary efficiently dominatable graphs. We also give a decomposition theorem for hereditary efficiently dominatable graphs and examine some algorithmic aspects of such graphs. In particular, we give a polynomial time algorithm for finding an efficient dominating set (if one exists) in a class of graphs properly containing the class of hereditary efficiently dominatable graphs by reducing the problem to the maximum weight independent set problem in claw-free graphs.
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1. INTRODUCTION

The concept of an efficient dominating set in a graph (also known as a perfect code, 1-perfect code, perfect dominating set, perfect independent dominating set) was introduced by Biggs [3] as a generalization of the notion of a perfect error-correcting code in coding theory. Given a (simple, finite, undirected) graph $G = (V, E)$, we say that a vertex *dominates* itself and each of its neighbors. An *efficient dominating set* in G is a subset of vertices $D \subseteq V$ such that every vertex $v \in V$ is dominated by precisely one vertex from D. Efficient domination has several interesting applications in coding theory and resource allocation of parallel processing systems [3, 38, 40].

We say that a graph is *efficiently dominatable* (ED) if it contains an efficient dominating set. Not all graphs are ED. See Figure 1 for some small examples of non-ED graphs: the bull, the fork, and the 4-cycle C_4. All paths are ED, and a cycle C_k on k vertices is ED if and only if k is divisible by 3.

Bange et al. [1] showed that if a graph G has an efficient dominating set, then all efficient dominating sets of G have the same cardinality, which equals the minimum cardinality of a dominating set of G. The problem of recognizing ED graphs is NP-complete even for restricted graph classes such as planar cubic graphs [36], bipartite graphs [48, 54], planar bipartite graphs [41], chordal bipartite graphs [41], chordal graphs [48, 54], and line graphs of planar bipartite graphs of maximum degree 3 [4]. On the other hand, recognizing ED graphs can be done in polynomial time for several graph classes, including trees [1, 30], block graphs [54], series-parallel graphs [53], interval graphs [11, 13, 35, 37], circular-arc graphs [13, 35], cocomparability graphs [10, 11], bipartite permutation graphs [41], permutation graphs [38], distance-hereditary graphs [41], trapezoid graphs [38, 39], split graphs [12], and AT-free graphs [8]. Trees on n vertices with maximum number of efficient dominating sets were characterized in [7], while efficient dominating sets in Sierpiński graphs were considered in [33].

The difficulty of characterizing ED graphs in general is perhaps related to the fact that they do not form a hereditary graph class (that is, a class of graphs closed under vertex deletions): adding a dominating vetrtex to *any* graph results in an ED graph. Motivated by this fact, we introduce and investigate in this article the *hereditary ED* graphs:

Definition 1. *A graph G is* hereditary ED *if every induced subgraph of G is ED.*

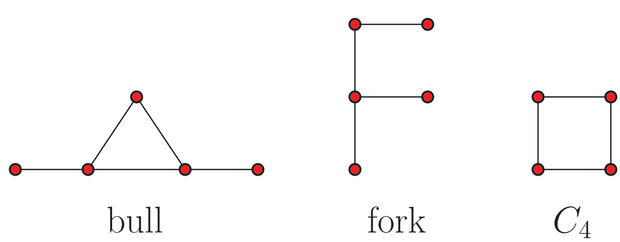

FIGURE 1. Some small non-ED graphs.

It follows directly from the definition that the set HED of hereditary ED graphs forms a hereditary graph class; hence, with every graph $G \in$ HED, the class HED contains all induced subgraphs of G. The family of hereditary graph classes is of particular interest, since hereditary (and only hereditary) classes admit a uniform description in terms of forbidden induced subgraphs. For a set $\mathcal{F}$ of graphs, we say that a graph G is *$\mathcal{F}$-free* if it does not contain an induced subgraph isomorphic to a member of $\mathcal{F}$. Given a hereditary class $\mathcal{H}$, the set $\mathcal{F}$ of all graphs F with the property that $F \notin \mathcal{H}$ but $H \in \mathcal{H}$ for every proper induced subgraph H of F is said to be the set of forbidden induced subgraphs for $\mathcal{H}$, and $\mathcal{H}$ is precisely the class of $\mathcal{F}$-free graphs. Every hereditary class of graphs can be characterized as being $\mathcal{F}$-free for some (finite or infinite) family $\mathcal{F}$, and for many interesting classes this characterization is known. Such characterizations can be useful for establishing inclusion relations among hereditary graph classes; perhaps the most famous characterization of this kind was obtained for perfect graphs in the famous Strong Perfect Graph Theorem conjectured by Berge in 1961 [2] and proved by Chudnovsky, Robertson, Seymour, and Thomas in 2006 [17].

There are also theorems that elucidate the structure of graphs in a certain hereditary class by showing that every graph in the class either belongs to one of a few basic classes (in which case it has a prescribed and relatively transparent structure) or it has one of a set of prescribed structural faults, along which it can be decomposed in a useful way. Several such decomposition results were obtained in recent years, including those for Meyniel graphs [9], perfect graphs [17], cap-free graphs [21], universally signable graphs [22, 25], even-hole-free graphs [23], graphs without odd holes, parachutes or proper wheels [20], odd-hole-free graphs [24], and (diamond, even hole)-free graphs [34]. Few results of a stronger type are also known, in which the decomposition can also be reversed in the sense that a graph is in the class if and only if it can be constructed by gluing basic graphs along the decompositions prescribed. Such *composition* results are known, among others, for chordal graphs [28], claw-free graphs [18], graphs with no cycle with a unique chord [50], bull-free graphs [15–16], and graphs of separability at most two [19]. Decomposition results often have nice algorithmic consequences and provide means for obtaining bounds on certain graph parameters in terms of others.

Besides purely graph theoretical motivations for the study of hereditary ED graphs, this notion also finds applications in a practical setting. For example (following [40]), consider a parallel computer network modeled by a graph with edges representing communication links between processing units. There is a limited amount of a certain resource that the units must be able to access for their work. If the underlying graph contains an efficient dominating set, then such a set can be used to distribute the resource to the processors in the most efficient way—in the sense that every processor will be able to access the resource either directly, or via a unique processor connected to it with a communication link. If, in addition, the underlying graph is hereditary ED, the network will be very robust with respect to this efficient way of distributing the resource: in the case of malfunctioning of *any* subset of processors, we will still be able to obtain an efficient distribution of the resource to the remaining processors. Moreover, as we will show in Section 5, an efficient dominating set that can be used to realize such a distribution can be found in polynomial time.

Since the bull, the fork, and cycles of the form C_{3k+1} or C_{3k+2} are not ED, the class of hereditary ED graphs is contained in the class of (bull, fork, C_{3k+1}, C_{3k+2})-free graphs. We show in this article that the converse inclusion holds as well. This result will be obtained in Section 4 as a corollary to a structural characterization of (bull, fork, C_4)-free graphs (Section 3), which implies that every (bull, fork, C_4)-free graph can be built from paths and cycles by applying a sequence of the following operations: disjoint union of two graphs, vertex duplication, addition of a dominating vertex, raft expansion, and semi-raft expansion (see Section 2 for definitions). A similar structural result is obtained also for hereditary ED graphs. Finally, we will outline, in Section 5, two polynomial time algorithms: (i) a robust algorithm that finds either an efficient dominating set in the input graph, or a certificate for the fact that the graph is not hereditary ED, and (ii) an algorithm for finding the maximum number of vertices that can be efficiently dominated, for graphs in a class of graphs properly containing (bull, fork)-free graphs. While the former result is obtained as a consequence of the decomposition theorem, the latter one employs an entirely different approach: it reduces the problem to the maximum weight independent set problem in claw-free graphs.

2. PRELIMINARY DEFINITIONS

The structure theorem for (bull, fork, C_4)-free graphs, developed in Section 3, will rely on certain basic graphs and certain operations that build larger graphs from smaller ones. We introduce the necessary definitions in this section.

For a graph G and a set $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced by X. For two disjoint sets $X, Y \subseteq V(G)$, we denote by $G[X, Y]$ *the bipartite subgraph of G between X and Y*, that is, the graph $(X \cup Y, \{xy \in E(G) : x \in X, y \in Y\})$. For a vertex $x \in V(G)$, we denote by $N(x)$ the neighborhood of x, i.e., the set of vertices adjacent to x, and by $N[x] := N(x) \cup \{x\}$ the closed neighborhood of x. A dominating vertex in a graph G is a vertex v such that $N[v] = V(G)$. We say that a vertex v *dominates* a set X if v is adjacent to every vertex in X. We denote by P_k a path on k vertices, and by $2K_2$ the graph consisting of two disjoint copies of K_2. A graph is *chordal* if it has no induced cycle of order 4 or more. To simplify notation, we will say that a graph is $(F_1, \dots, F_k)$-free if no induced subgraph of it is isomorphic to a member of $\mathcal{F} = \{F_1, \dots, F_k\}$. Also, we will say that a graph G is (bull, fork, C_{3k+1}, C_{3k+2})-free if it is $\mathcal{F}$-free for $\mathcal{F} = \{$bull, fork$\} \cup \bigcup_{k\geq 1}\{C_{3k+1}, C_{3k+2}\}$. For terms left undefined, we refer the reader to [27].

Duplicating a vertex v in a graph G means adding to G a new vertex and making it adjacent to v and to all neighbors of v. Duplicated vertices are also known in the literature as *true twins*.

In order to describe the operations of a raft expansion and a semiraft expansion, we need to define two particular graph families: rafts and semirafts. For $n \geq 0$, a *raft* R_n is a graph consisting of two disjoint cliques on $n + 1$ vertices each, say $X = \{x_0, x_1, \dots, x_n\}$ and $Y = \{y_0, y_1, \dots, y_n\}$ together with additional edges between $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ such that for every $1 \leq i, j \leq n$, vertex x_i is adjacent to vertex y_j if and only if $j \geq i$. We say that X and Y are *parts* of the raft, and x_0 and y_0 are the *tips* of the raft. A raft is *nontrivial* if $n \geq 1$. The *trivial* raft R_0 consists of two isolated vertices. See Figure 2 for examples of rafts.

For $n \geq 1$, a *semiraft* S_n is a graph consisting of two disjoint cliques on n vertices each, say $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$ together with additional edges between $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ such that for every $1 \leq i, j \leq n$, vertex x_i is adjacent to vertex y_j if and only if $j \geq i$. We say that X and Y are *parts* of the semiraft, and x_1 and

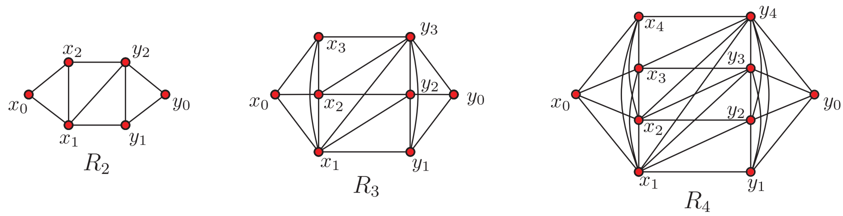

FIGURE 2. Rafts R_2, R_3, and R_4.

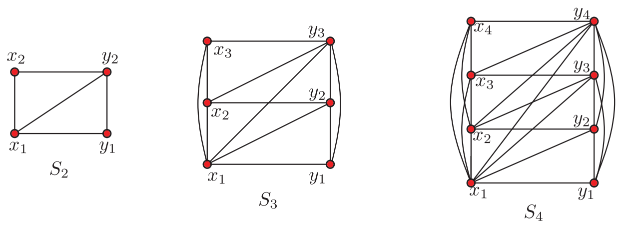

FIGURE 3. Semirafts S_2, S_3, and S_4.

y_n are the *tips* of the semiraft. A semiraft is *nontrivial* if $n \geq 2$. The *trivial* semiraft S_1 consists of two adjacent vertices. See Figure 3 for examples of semirafts.

Given a graph G and two nonadjacent vertices x, y in G, a *raft expansion* (with respect to x, y) is the operation that replaces G with a graph G' such that:

- $V(G') = (V(G) \setminus \{x, y\}) \cup (X \cup Y)$ such that
- $G'[X \cup Y]$ is a raft with parts X and Y and
- $E(G') = \{uv \in E(G) : \{u, v\} \cap \{x, y\} = \emptyset\} \cup \{ux' : x' \in X, ux \in E(G)\} \cup \{uy' : y' \in Y, uy \in E(G)\} \cup E(G'[X \cup Y])$, where $E(G'[X \cup Y])$ are the edges of $G'[X \cup Y]$.

We say that $G'[X \cup Y]$ forms a *raft in* G'. Notice that G is an induced subgraph of G'.

Given a graph G and two adjacent vertices x, y in G, a *semiraft expansion* (with respect to x, y) is the operation that replaces G with a graph G' such that:

- $V(G') = (V(G) \setminus \{x, y\}) \cup (X \cup Y)$ such that
- $G'[X \cup Y]$ is a semiraft with parts X and Y and
- $E(G') = \{uv \in E(G) : \{u, v\} \cap \{x, y\} = \emptyset\} \cup \{ux' : x' \in X, ux \in E(G)\} \cup \{uy' : y' \in Y, uy \in E(G)\} \cup E(G'[X \cup Y])$, where $E(G'[X \cup Y])$ are the edges of $G'[X \cup Y]$.

We say that $G'[X \cup Y]$ forms a *semiraft in* G'. Notice that G is an induced subgraph of G'.

In our proof of Theorem 2 below, we will often use the following fact about $2K_2$-free bipartite graphs (see, e.g., [32]):

Proposition 1. *Let G be a $2K_2$-free bipartite graph with a given bipartition $V(G) = A \cup B$ of its vertex set. Then, vertices in A can be linearly ordered with respect to inclusion of their neighborhoods. In particular, if B has no isolated vertices, then there exists a vertex in A that is adjacent to all vertices in B.*

Proof. For the fact that vertices in A can be linearly ordered with respect to inclusion of their neighborhoods, see, e.g., [32]. If in addition every vertex in B has a neighbor in A, we can write $A = \{v_1, \dots, v_k\}$ where $N(v_1) \subseteq N(v_2) \subseteq \cdots \subseteq N(v_k)$, and we easily see that $v_k \in A$ dominates B. ■

The $2K_2$-free bipartite graphs are also known in the literature as *chain graphs* (defined as bipartite graphs such that for each color class the neighborhoods of the nodes in that color class can be ordered linearly w.r.t. inclusion [52]). It is worth mentioning their close connection to rafts or semirafts: the complement of a chain graph either has duplicated vertices or is a raft or semiraft.

3. A STRUCTURE THEOREM FOR (BULL, FORK, C_4)-FREE GRAPHS

In this section, we prove the following structural result about (bull, fork, C_4)-free graphs:

Theorem 2. *Let G be a (bull, fork, C_4)-free graph. Then, G can be built from paths and cycles by applying a sequence of the following operations:*

- *disjoint union of two graphs,*
- *vertex duplication,*
- *addition of a dominating vertex,*
- *raft expansion,*
- *semiraft expansion.*

Proof. Let G be a counterexample minimizing $|V(G)|$. Then, G is a connected graph that is neither a path nor a cycle, has no dominating or duplicated vertices, and is not the result of a raft expansion or a semiraft expansion.

We split the proof into two cases, according to whether G contains a *hole* (an induced cycle of order 4 or more) or not.

Case 1: G contains a hole.

Let $C = (v_0, v_1, \dots, v_k)$ be a shortest hole in G. By the C_4-freeness of G, we have $k \geq 4$.

Claim 1. *For every vertex $v \in V(G) \setminus V(C)$, one of the following occurs:*

(1) *v has no neighbors on C,*
(2) *v dominates C,*
(3) *the neighborhood of v on C consists of precisely three consecutive vertices on C.*

Proof. Let $v \in V(G) \setminus V(C)$ be a vertex that has a neighbor on C but does not dominate C. Then, there exist two adjacent vertices on $C = (v_0, v_1, \dots, v_k)$, say v_0 and v_1, such that v is adjacent to v_1 but not to v_0. Then, v must be adjacent to either v_2 or v_3, since otherwise the set $\{v, v_0, v_1, v_2, v_3\}$ would induce a fork in G. It cannot be adjacent only to v_3, since otherwise the set $\{v, v_1, v_2, v_3\}$ would induce a C_4 in G. It cannot be adjacent only to v_2, since otherwise the set $\{v, v_0, v_1, v_2, v_3\}$ would induce a bull in G. Therefore, v is adjacent to both v_2 and v_3. Let $r = \max\{j : 1 \leq j \leq k, vv_j \in E(G)\}$. Since $vv_3 \in E(G)$, $r \geq 3$. If $r > 3$, then the subgraph of G induced by $\{v, v_r, v_{r+1}, \dots, v_k, v_0\}$ forms a hole in G shorter than C, contradicting the choice of C. Therefore, $r = 3$ and the neighborhood of v on C consists of precisely three consecutive vertices on C. ■

Claim 2. *There is no vertex $v \in V(G) \setminus V(C)$ such that the neighborhood of v on C consists of precisely three consecutive vertices on C.*

Proof. Suppose for a contradiction that there exists $v \in V(G) \setminus V(C)$ such that $N(v) \cap V(C) = \{v_1, v_2, v_3\}$. Then, since G has no duplicated vertices, one of the sets $N[v] \setminus N[v_2]$ and $N[v_2] \setminus N[v]$ is nonempty. We may assume without loss of generality that there exists a vertex $u \in N[v_2] \setminus N[v]$. Vertex u cannot be adjacent to both v_1 and v_3, since otherwise the set $\{u, v_1, v, v_3\}$ would induce a C_4 in G. Thus, we may assume without loss of generality that $N(u) \cap V(C) = \{v_0, v_1, v_2\}$. However, applying Claim 1 to a shortest hole $C' = (v_0, v_1, v, v_3, v_4, \ldots, v_k)$, we conclude that $N(u) \cap V(C') = \{v_k, v_0, v_1\}$, which is a contradiction with the fact that $N(u) \cap V(C) = \{v_0, v_1, v_2\}$. ■

Claim 3. *No vertex $v \in V(G) \setminus V(C)$ dominates C.*

Proof. Suppose for a contradiction that there exists a vertex that dominates C. By the C_4-freeness of G, every two vertices that dominate C are adjacent. Let K denote the clique of vertices that dominate C.

First, we show that every vertex outside K has a neighbor in K. Suppose not. Since G is connected, there exist a vertex x at distance 2 from K. Let (x, y, v) be a path connecting x to K. If y is not in C then the set $\{x, y, v, v_1, v_3\}$ induces a fork in G, a contradiction. Hence, y belongs to C and by Claim 1, the neighborhood of x on C consists of precisely three consecutive vertices on C, say $\{v_i, v_{i+1}, v_{i+2}\}$. Hence, the set $\{x, v_i, v, v_{i+1}\}$ induces a C_4 in G, a contradiction.

Let $W := V(G) \setminus (V(C) \cup K)$, and let $H = G[K, W]$ be the bipartite subgraph of G between K and W. Then, H is $2K_2$-free, since otherwise, if (v, w, v', w') induce a $2K_2$ in H (with $v, v' \in K, w, w' \in W$), then G would contain either a C_4 (on the vertices $\{v, w, w', v'\}$) if $ww' \in E(G)$ or a bull (on the vertices $\{v, w, w', v', v_0\}$) if $ww' \notin E(G)$. Therefore, by Proposition 1 and the fact that every vertex outside K has a neighbor in K, there exists a vertex v in K that dominates W. Such a vertex dominates G, contrary to the assumption that G has no dominating vertices. ■

Therefore, since G is connected, it follows that $G = C$, so G is a cycle, which is a contradiction. This completes the proof for Case 1.

Case 2: G is chordal.

Let $P = (v_1, \ldots, v_k)$ be a longest induced path in G. We may assume that $k \geq 4$ since if $k \leq 3$ then G is a connected (P_4, C_4)-free graph and therefore it contains a dominating vertex [51], which is impossible.

Claim 4. *For every vertex $v \in V(G) \setminus V(P)$, one of the following occurs:*

(1) *v has no neighbors on P,*
(2) *v dominates P,*
(3) *the neighborhood of v on P consists of two consecutive vertices on P, one of which is an endpoint of P (such a vertex v will be called an end neighbor of P),*
(4) *the neighborhood of v on P consists of three consecutive vertices on P (such a vertex v will be called a central neighbor of P).*

Proof. Let $v \in V(G) \setminus V(P)$ be a vertex that has a neighbor on P but does not dominate P.

Suppose first that v is adjacent to an endpoint of P, say v_1. Then v must be adjacent to v_2 (otherwise, by the maximality of P, v must have another neighbor on P which would create a hole in G). Suppose that v is not adjacent to v_3. Then, since G is chordal, v cannot be adjacent to any other vertex on P except v_1 and v_2, and so v is an end vertex. Now suppose that v is adjacent to v_3. If $k = 4$, then v is either a central neighbor of P or dominates P. Suppose now that $k \geq 5$, that v does not dominate P, and that $N(v) \cap V(P) \neq \{v_1, v_2, v_3\}$. Let $r = \max\{j : 3 \leq j \leq k-1, \{v_1, v_2, \ldots, v_j\} \subseteq N(v)\}$. If there exists a vertex in $\{v_{r+2}, v_{r+3}, \ldots, v_k\} \cap N(v)$, then G contains a hole on the vertex set $\{v, v_r, v_{r+1}, \ldots, v_\ell\}$, where $v_\ell \in \{v_{r+2}, v_{r+3}, \ldots, v_k\} \cap N(v)$ is the vertex with the minimum value of ℓ. If $N(v) \cap V(P) = \{v_1, \ldots, v_r\}$, then $r \geq 4$ and G contains a bull on the vertex set $\{v_1, v, v_{r-1}, v_r, v_{r+1}\}$. In either case, we reach a contradiction.

Now, suppose that v is not adjacent to any of the endpoints on P, but is adjacent to some v_r with $1 < r < k$. Then v must be adjacent to either v_{r-1} or to v_{r+1} since otherwise G would contain either a hole (if v is adjacent to one of v_{r-2}, v_{r+2}) or a fork (otherwise). We may assume without loss of generality that v is adjacent to v_{r-1} (otherwise, simply replace r with $r+1$). Then, v must be adjacent to either v_{r-2} or to v_{r+1} since otherwise G would contain a bull on the vertex set $\{v, v_{r-2}, v_{r-1}, v_r, v_{r+1}\}$. Again, we may assume without loss of generality that v is adjacent to v_{r+1} (otherwise, replace r with $r-1$). Suppose that v is adjacent to some v_j with $2 \leq j \leq r-2$. Take j as small as possible. If $j = r-2$, then G contains a bull on the vertex set $\{v_{r-3}, v_{r-2}, v_{r-1}, v, v_{r+1}\}$. If $j \leq r-3$ then G contains a fork on the vertex set $\{v_{j-1}, v_j, v, v_{r-1}, v_{r+1}\}$. In either case, we reach a contradiction, implying that v is not adjacent to any v_j with $j < r-1$. By symmetry, v is also not adjacent to any v_j with $j > r+1$. Hence, v is a central neighbor of P. ■

Claim 5. *If $k \geq 5$, then P has no central neighbors.*

Proof. Let $k \geq 5$, and suppose for a contradiction that there exists a vertex $v \in V(G) \setminus V(P)$ such that $N(v) \cap V(P) = \{v_{r-1}, v_r, v_{r+1}\}$ for some $1 < r < k$. Since G has no duplicated vertices, we may assume that there exists a neighbor of v, say w, that is not adjacent to v_r. Then, w cannot be adjacent to both v_{r-1} and v_{r+1}, since otherwise G would contain a C_4. In particular, w cannot dominate P or $P' = (v_1, \ldots, v_{r-1}, v, v_{r+1}, \ldots, v_k)$. Since P' is a longest induced path in G, Claim 4 implies that w is either a central neighbor of P' or an end neighbor of P'. In particular, w is adjacent to precisely one of v_{r-1}, v_{r+1}, say to v_{r-1}. But now, G contains either a C_5 on the vertex set $\{w, v_{r-1}, v_r, v_{r+1}, v_{r+2}\}$ (if w is adjacent to v_{r+2}) or a bull on the vertex set $\{w, v, v_r, v_{r+1}, v_{r+2}\}$ (otherwise). In either case, we reach a contradiction. ■

Claim 6. *If $k \geq 5$ then P has no end neighbors.*

Proof. Let $k \geq 5$, and suppose for a contradiction that there exists a vertex $v \in V(G) \setminus V(P)$ such that $N(v) \cap V(P) = \{v_1, v_2\}$. Since G has no duplicated vertices, we may assume that there exists a neighbor of v, say w, that is not adjacent to v_1. Let $P' = (v, v_2, \ldots, v_k)$. Then P' is a longest induced path of G, and we can apply Claims 4 and 5 to w and P' to conclude that $N(w) \cap V(P') = \{v, v_2\}$. However, this implies that $N(w) \cap V(P) = \{v_2\}$, contrary to Claim 4. ■

Claim 7. *If $k \geq 5$, no vertex dominates P.*

Proof. Suppose for a contradiction that there exists a vertex that dominates P. By the C_4-freeness of G, every two vertices that dominate P are adjacent. Let K denote the clique of vertices that dominate P. Notice that every vertex outside K is either on P or has no neighbor on P. First, we show that every vertex outside K has a neighbor in K. Suppose not. Since G is connected, there exist a vertex x at distance two from K. Denoting by (x, y, v) a path connecting x to K, we see that the set $\{x, y, v, v_1, v_3\}$ induces a fork in G, a contradiction.

Let $W := V(G) \setminus (V(P) \cup K)$, and let $H = G[K, W]$ be the bipartite subgraph of G between K and W. Then, H is $2K_2$-free, since if $\{v, w, v', w'\}$ induce a $2K_2$ in H (with $v, v' \in K$, $w, w' \in W$), then G would contain either a C_4 (on the vertices $\{v, w, w', v'\}$) if $ww' \in E(G)$ or a bull (on the vertices $\{v, w, w', v', v_1\}$) if $ww' \notin E(G)$. Therefore, by Proposition 1 and the fact that every vertex outside K has a neighbor in K, there exists a vertex v in K that dominates W. Such a vertex dominates G, contrary to the assumption that G has no dominating vertices. ■

Combining Claims 4, 5, 6, and 7, and the fact that G is connected, we conclude that G is a path if $k \geq 5$, which is a contradiction.

Hence $k = 4$. Using Claim 4, we can partition the vertices in $N(P)$ into five sets:

$$N(P) = V_{12} \cup V_{123} \cup V_{234} \cup V_{34} \cup V_{1234}\,,$$

where:

- V_{12} is the set of "left" end neighbors of P: $V_{12} = \{v \in V(G) : N(v) \cap V(P) = \{v_1, v_2\}\}$,
- V_{123} is the set of "left" central neighbors of P: $V_{123} = \{v \in V(G) : N(v) \cap V(P) = \{v_1, v_2, v_3\}\}$,
- V_{234} is the set of "right" central neighbors of P: $V_{234} = \{v \in V(G) : N(v) \cap V(P) = \{v_2, v_3, v_4\}\}$,
- V_{34} is the set of "right" end neighbors of P: $V_{34} = \{v \in V(G) : N(v) \cap V(P) = \{v_3, v_4\}\}$,
- V_{1234} is the set of vertices that dominate P: $V_{1234} = \{v \in V(G) : N(v) \cap V(P) = \{v_1, v_2, v_3, v_4\}\}$.

Now we introduce certain partitions of each of the sets $V_{12}, V_{123}, V_{234}, V_{34}$ into two parts.

Claim 8. *Let*

- $V_{12}^+ := \{v \in V_{12}$: *there exists a vertex* $w \in V_{234}$ *such that* $vw \in E(G)\}$,
- $V_{12}^- := \{v \in V_{12}$: *there exists a vertex* $w \in V_{1234}$ *such that* $vw \notin E(G)\}$.

Then, $V_{12} = V_{12}^+ \cup V_{12}^-$ *and* $V_{12}^+ \cap V_{12}^- = \emptyset$.

Proof. Let $v \in V_{12}$. Since G has no duplicated vertices, there exists a neighbor of v that is not adjacent to v_1 or a neighbor of v_1 that is not adjacent to v. Suppose that there exists a neighbor w of v_1 that is not adjacent to v. Claim 4 implies that $w \in V_{12} \cup V_{123} \cup V_{1234}$. Applying Claim 4 to w and the path (v, v_2, v_3, v_4), we conclude that $w \in V_{1234}$. Suppose now that there exists a neighbor w of v that is not adjacent to v_1. Claim 4 implies that $w \in V_{234} \cup V_{34}$. Applying Claim 4 to w and the path (v, v_2, v_3, v_4), we conclude that $w \in V_{234}$.

This shows that every vertex $v \in V_{12}$ has a neighbor in V_{234} or a nonneighbor in V_{1234}. To see that it cannot have both at the same time, suppose for a contradiction that there exists a $v \in V_{12}$ with a neighbor w in V_{234} and a nonneighbor w' in V_{1234}. To avoid an induced C_4 on vertices $\{w, v_2, w', v_4\}$, we conclude that $ww' \in E(G)$. But now, a C_4 arises on the vertices $\{w, w', v, v_1\}$, a contradiction. ■

By symmetry with Claim 8, we obtain the following claim.

Claim 9. *Let*

- $V_{34}^{+} := \{v \in V_{34}$: *there exists a vertex* $w \in V_{123}$ *such that* $vw \in E(G)\}$,
- $V_{34}^{-} := \{v \in V_{34}$: *there exists a vertex* $w \in V_{1234}$ *such that* $vw \notin E(G)\}$.

Then, $V_{34} = V_{34}^{+} \cup V_{34}^{-}$ *and* $V_{34}^{+} \cap V_{34}^{-} = \emptyset$.

Claim 10. *Let*

- $V_{123}^{+} := \{v \in V_{123}$: *there exists a vertex* $w \in V_{34}$ *such that* $vw \in E(G)\}$,
- $V_{123}^{-} := \{v \in V_{123}$: *there exists a vertex* $w \in V_{234}$ *such that* $vw \notin E(G)\}$.

Then, $V_{123} = V_{123}^{+} \cup V_{123}^{-}$ *and* $V_{123}^{+} \cap V_{123}^{-} = \emptyset$.

Proof. Let $v \in V_{123}$. Since G has no duplicated vertices, there exists a neighbor of v that is not adjacent to v_2 or a neighbor of v_2 that is not adjacent to v. Suppose that there exists a neighbor w of v_2 that is not adjacent to v. Then $w \in V_{12} \cup V_{123} \cup V_{234} \cup V_{1234}$, but out of these four possibilities, only $w \in V_{234}$ does not contradict Claim 4 applied to w and the path (v_1, v, v_3, v_4). Therefore $w \in V_{234}$.

Suppose now that there exists a neighbor w of v that is not adjacent to v_2. Claim 4, applied to w and the path (v_1, v, v_3, v_4), implies that $N(w) \cap \{v_1, v, v_3, v_4\} \in \{\{v_1, v\}, \{v_1, v, v_3\}, \{v, v_3, v_4\}, \{v_1, v, v_3, v_4\}\}$. Out of these four possibilities, only the third one does not contradict Claim 4, applied to w and P. Therefore $w \in V_{34}$.

This shows that every vertex $v \in V_{123}$ has a neighbor in V_{34} or a nonneighbor in V_{234}. To see that it cannot have both at the same time, suppose for a contradiction that there exists a $v \in V_{123}$ with a neighbor w in V_{34} and a nonneighbor w' in V_{234}. To avoid an induced C_4 on vertices $\{w, v, v_2, w'\}$, we conclude that $ww' \notin E(G)$. But now, a C_5 arises on the vertices $\{w, v, v_2, w', v_4\}$, a contradiction. ■

By symmetry with Claim 10, we obtain the following claim.

Claim 11. *Let*

- $V_{234}^{+} := \{v \in V_{234}$: *there exists a vertex* $w \in V_{12}$ *such that* $vw \in E(G)\}$,
- $V_{234}^{-} := \{v \in V_{234}$: *there exists a vertex* $w \in V_{123}$ *such that* $vw \notin E(G)\}$.

Then, $V_{234} = V_{234}^{+} \cup V_{234}^{-}$ *and* $V_{234}^{+} \cap V_{234}^{-} = \emptyset$.

In the following claim, we collect statements about the forced adjacencies and nonadjancencies between vertices of $N(P)$.

Claim 12. *The following holds:*

(a) *Each of the sets* $V_{12}, V_{123}, V_{234}, V_{34}, V_{1234}$ *is a clique.*
(b) *For every two (distinct) sets* $U, W \in \{V_{12}, V_{123}, V_{234}, V_{34}, V_{1234}\}$*, the bipartite subgraph of* G *between* U *and* W *is* $2K_2$*-free.*

(c) *Every vertex in V_{12} is adjacent to every vertex in V_{123}. Similarly, every vertex in V_{34} is adjacent to every vertex in V_{234}.*
(d) *Every vertex of V_{1234} is adjacent to every vertex in $V_{123} \cup V_{234}$.*
(e) *Every vertex of V_{12} is nonadjacent to every vertex in V_{34}.*
(f) *Every vertex of V_{1234} is adjacent to every vertex in $V_{12}^{+} \cup V_{34}^{+}$.*
(g) *Every vertex of V_{123}^{+} is adjacent to every vertex of V_{234}.*
(h) *Every vertex of V_{234}^{+} is adjacent to every vertex of V_{123}.*
(i) *Every vertex of V_{12}^{-} is nonadjacent to every vertex of V_{234}.*
(j) *Every vertex of V_{34}^{-} is nonadjacent to every vertex of V_{123}.*
(k) *Every vertex of V_{123}^{-} is nonadjacent to every vertex of V_{34}.*
(l) *Every vertex of V_{234}^{-} is nonadjacent to every vertex of V_{12}.*

Proof. We only give the sketch of the proof, as the reader should have no problems verifying the details.

(a) For $V_{123}, V_{234}, V_{1234}$, this follows from the C_4-freeness of G. For V_{12} and V_{34}, this follows from the fork-freeness of G.
(b) follows (a) and the C_4-freeness of G.
(c) follows from the bull-freeness of G.
(d)–(e) follow from the C_4-freeness of G.
(f)–(l) follow from Claims 8 to 11. ■

Let us denote by R the set $V(G) \setminus (V(P) \cup N(P))$.

Claim 13. *Every vertex in R has a neighbor in V_{1234}.*

Proof. Due to the maximality of P, no vertex from R has a neighbor in $V_{12} \cup V_{34}$.
Next, due to the fork-freeness of G, no vertex from R has a neighbor in $V_{123} \cup V_{234}$.
Suppose for a contradiction that there exists a vertex in R that has no neighbor in V_{1234}. Then G contains an induced P_3 of the form (r', r, v), where $r', r \in R$ and $v \in V_{1234}$. However, this would produce a fork on the vertex set $\{r', r, v, v_1, v_3\}$, a contradiction. Therefore, every vertex in R has a neighbor in V_{1234}. ■

Claim 14. *No vertex in V_{1234} has a neighbor in R.*

Proof. Suppose for a contradiction that a vertex $v \in V_{1234}$ has a neighbor in R.
It follows from the C_4- and bull-freeness of G that the bipartite subgraph $G[V_{1234}, R]$ is $2K_2$-free. Hence, by Claim 13 and Proposition 1, there exists a vertex v in V_{1234} that dominates R. Since G has no dominating vertices, v has a nonneighbor w which, by Claim 12, belongs to the set $V_{12}^{-} \cup V_{34}^{-}$. Without loss of generality, we may assume that $w \in V_{12}^{-}$. By the definition of the set V_{12}^{-}, vertex w has a nonneighbor, say r, in the set V_{1234}. But now, G contains a fork induced on the vertex set $\{r, v, v_1, w, v_4\}$, a contradiction. ■

It follows from Claims 13 and 14 that $R = \emptyset$.

Claim 15. *Let*

- $V'_{1234} := \{v \in V_{1234} : v$ *is adjacent to every vertex of* $V_{12}^{-}\}$.
- $V''_{1234} := \{v \in V_{1234} : v$ *is adjacent to every vertex of* $V_{34}^{-}\}$,

Then, $V_{1234} = V'_{1234} \cup V''_{1234}$ *and* $V'_{1234} \cap V''_{1234} = \emptyset$.

Proof. Observe that by the bull-freeness of G, every vertex of V_{1234} is adjacent either to every vertex of V_{12}^- or to every vertex of V_{34}^-. On the other hand, since G has no dominating vertices, no vertex in V_{1234} can be adjacent to every vertex of $V_{12}^- \cup V_{34}^-$. ■

Claim 16. *The graph $G[V_{123}^+ \cup V_{34}^+]$ is either empty or forms a semiraft in G with parts V_{123}^+ and V_{34}^+.*

Proof. Suppose $G[V_{123}^+ \cup V_{34}^+]$ is nonempty. By the definition of the set V_{123}^+ and Claim 12(j), every vertex of V_{123}^+ has a neighbor in V_{34}^+. Similarly, every vertex of V_{34}^+ has a neighbor in V_{123}^+. Thus, neither V_{123}^+ nor V_{34}^+ is empty.

By Claim 12(b), the graph $H = G[V_{123}^+, V_{34}^+]$ is $2K_2$-free. Therefore, the vertices of V_{123}^+ can be linearly ordered with respect to inclusion of their neighborhoods in V_{34}^+. Let $V_{123}^+ = \{x_1, \dots, x_n\}$ such that $N_H(x_1) \subseteq N_H(x_2) \subseteq \cdots \subseteq N_H(x_n)$. By Claim 12(a), each of the sets V_{123}^+ and V_{34}^+ is a clique. Since G has no duplicated vertices, all the inclusions are proper: $N_H(x_1) \subset N_H(x_2) \subset \cdots \subset N_H(x_n)$. Moreover, denoting $N_H(x_0) = \emptyset$, we have, for the same reason,

$$|N_H(x_i) \setminus N_H(x_{i-1})| = 1$$

for every $1 \leq i \leq n$: any two vertices in some $N_H(x_{i+1}) \setminus N_H(x_i)$ would be duplicates of each other in G.

Using the fact that every vertex of V_{123}^+ has a neighbor in V_{34}^+ and vice versa, we conclude that $|V_{34}^+| = |V_{123}^+|$ and that $G[V_{123}^+ \cup V_{34}^+]$ forms a semiraft in G with parts V_{123}^+ and V_{34}^+. ■

By symmetry with Claim 16, we obtain the following claim.

Claim 17. *The graph $G[V_{12}^+ \cup V_{234}^+]$ is either empty or forms a semiraft in G with parts V_{12}^+ and V_{234}^+.*

Claim 18. *The graph $G[V_{123}^- \cup V_{234}^-]$ is either empty or forms a raft in G with parts V_{123}^- and V_{234}^-.*

Proof. Suppose $G[V_{123}^- \cup V_{234}^-]$ is nonempty. By the definition of the set V_{123}^- and Claim 12(h), every vertex of V_{123}^- has a nonneighbor in V_{234}^-. Similarly, every vertex of V_{234}^- has a nonneighbor in V_{123}^-. Thus, neither V_{123}^- nor V_{234}^- is empty.

By Claim 12(b), the graph $H = G[V_{123}^-, V_{234}^-]$ is $2K_2$-free. Therefore, the vertices of V_{123}^- can be linearly ordered with respect to inclusion of their neighborhoods in V_{234}^-. Let $V_{123}^- = \{x_1, \dots, x_n\}$ such that $N_H(x_1) \subseteq N_H(x_2) \subseteq \cdots \subseteq N_H(x_n)$. By Claim 12(a), each of the sets V_{123}^- and V_{234}^- is a clique. Since G has no duplicated vertices, all the inclusions are proper: $N_H(x_1) \subset N_H(x_2) \subset \cdots \subset N_H(x_n)$. Moreover, $N_H(x_1) = \emptyset$ (since otherwise any neighbor of x_1 would have no nonneighbor in V_{123}^-), and

$$|N_H(x_i) \setminus N_H(x_{i-1})| = 1$$

for every $2 \leq i \leq n$: any two vertices in some $N_H(x_{i+1}) \setminus N_H(x_i)$ would be duplicates of each other in G.

Using the fact that every vertex of V_{123}^- has a nonneighbor in V_{234}^- and vice versa, we conclude that $|V_{234}^-| = |V_{123}^-|$ and that $G[V_{123}^- \cup V_{234}^-]$ forms a raft in G with parts V_{123}^- and V_{234}^-. ■

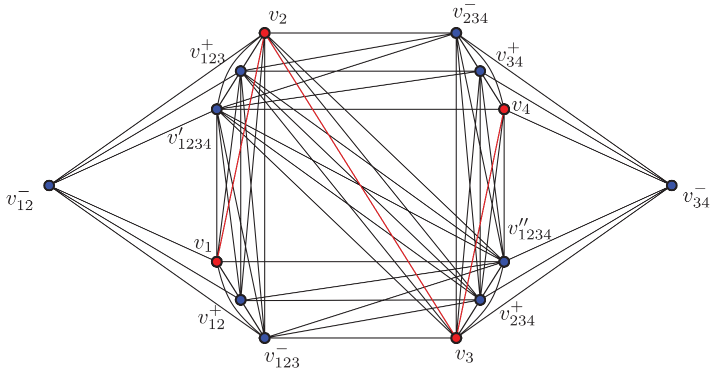

FIGURE 4. A 14-vertex graph extending the path (v_1, v_2, v_3, v_4).

The following two claims can be proved along similar lines as Claim 18.

Claim 19. *The graph $G[V^-_{12} \cup V''_{1234}]$ is either empty or forms a raft in G with parts V^-_{12} and V''_{1234}.*

Claim 20. *The graph $G[V^-_{34} \cup V'_{1234}]$ is either empty or forms a raft in G with parts V^-_{34} and V'_{1234}.*

In conjunction with the fact that G is not the result of a nontrivial raft expansion or a semiraft expansion, Claims 16–20 imply that each of the sets $V^+_{12}, V^-_{12}, V^+_{123}, V^-_{123}, V^+_{234}, V^-_{234}, V^+_{34}, V^-_{34}, V'_{1234}, V''_{1234}$ contains at most one vertex.

If each of these sets contains exactly one vertex, then G is isomorphic to the 14-vertex graph depicted in Figure 4 (where the unique elements of the above sets are denoted with lowercase letters, e.g., $V^+_{12} = \{v^+_{12}\}$, etc.).

Claim 21. *At least one of the sets $V^+_{123}, V^-_{234}, V^+_{34}$ is empty.*

Proof. Otherwise, G would contain a semiraft with parts $\{v_2, v^+_{123}\}$ and $\{v^-_{234}, v^+_{34}\}$. ■

By symmetry, we also obtain:

Claim 22. *At least one of the sets $V^+_{234}, V^-_{123}, V^+_{12}$ is empty.*

Claim 23. *At least one of the sets $V'_{1234}, V^+_{123}, V^+_{34}$ is empty.*

Proof. Otherwise, G would contain a semiraft with parts $\{v^+_{123}, v'_{1234}\}$ and $\{v_4, v^+_{34}\}$. ■

Claim 24. *$V^+_{123} = \emptyset$ if and only if $V^+_{34} = \emptyset$.*

Proof. If $V^+_{123} = \emptyset$ and $V^+_{34} \neq \emptyset$, then $N[v^+_{34}] = N[v_4]$, contrary to the fact that G has no duplicated vertices. Conversely, if $V^+_{34} = \emptyset$ and $V^+_{123} \neq \emptyset$, then $N[v^+_{123}] = N[v_2]$, which is again a contradiction. ■

By symmetry, we also obtain:

Claim 25. $V^+_{234} = \emptyset$ *if and only if* $V^+_{12} = \emptyset$.

Claim 26. $V^-_{123} = \emptyset$ *if and only if* $V^-_{234} = \emptyset$.

Proof. If $V^-_{123} = \emptyset$ and $V^-_{234} \neq \emptyset$, then $N[v^-_{234}] = N[v_3]$, contrary to the fact that G has no duplicated vertices. Conversely, if $V^-_{234} = \emptyset$ and $V^-_{123} \neq \emptyset$, then $N[v^-_{123}] = N[v_2]$, which is again a contradiction. ■

Claim 27. *At least one of the sets* V'_{1234} *and* V^-_{234} *is empty.*

Proof. If both sets V'_{1234} and V^-_{234} were nonemtpy then G would contain a semiraft with parts $\{v_2, v'_{1234}\} \cup V^+_{123}$ and $\{v^-_{234}, v_4\} \cup V34^+$ (independently of whether the sets V^+_{123} and V^+_{34} are both empty or both nonempty). ■

By symmetry, we also obtain:

Claim 28. *At least one of the sets* V''_{1234} *and* V^-_{123} *is empty.*

Claim 29. *If* $V'_{1234} = \emptyset$ *then* $V^-_{34} = \emptyset$.

Proof. If $V'_{1234} = \emptyset$ and $V^-_{34} \neq \emptyset$ then $N[v_4] = H[v^-_{34}]$, a contradiction with the fact that G has no duplicated vertices. ■

By a similar argument, we obtain:

Claim 30. *If* $V^-_{234} = \emptyset$ *then* $V^-_{12} = \emptyset$.

Claim 31. *If* $V'_{234} = \emptyset$ *then* $V^-_{12} = \emptyset$.

Proof. If $V'_{234} = \emptyset$ and $V^-_{12} \neq \emptyset$, then $N[v_2] = N[v^-_{12}]$, contrary to the fact that G has no duplicated vertices. ■

Claim 32. *At least one of the sets* V^-_{12} *and* V^-_{34} *is empty.*

Proof. Suppose that $V^-_{12} \neq \emptyset$. By Claim 31, the set V^-_{234} is nonempty. By Claim 27, the set V'_{1234} is empty. By Claim 29, the set V^-_{34} is empty. ■

Without loss of generality, we assume in what follows that $V^-_{34} = \emptyset$.

Claim 33. $V^-_{1234} = \emptyset$.

Proof. Otherwise, G would contain a dominating vertex v'_{1234}. ■

Claim 34. $V^-_{12} = \emptyset$.

Proof. Suppose that $V^-_{12} \neq \emptyset$. By Claim 31, $V^-_{234} \neq \emptyset$. By Claim 26, $V^-_{123} \neq \emptyset$. By Claim 30, $V''_{1234} \neq \emptyset$. However, the fact that V^-_{123} and V''_{1234} are both nonempty sets contradicts Claim 28. ■

Claim 35. $V''_{1234} = \emptyset$.

Proof. Otherwise, G would contain a dominating vertex v''_{1234}. ■

Claim 36. $V^-_{234} = \emptyset$.

Proof. Suppose that $V^-_{234} \neq \emptyset$. By Claim 26, we have $V^-_{123} \neq \emptyset$. By Claims 21 and 24, we have $V^+_{123} = V^+_{34} = \emptyset$. By Claims 22 and 25, we have $V^+_{234} = V^+_{12} = \emptyset$. But then, G is a raft with parts $\{v_1, v_2, v^-_{123}\}$ and $\{v_3, v_4, v^-_{234}\}$, a contradiction. ■

Claims 26 and 36 imply that $V^-_{123} = \emptyset$. In particular, G is a raft with parts $\{v_1, v_2\} \cup V^+_{12} \cup V^+_{123}$ (with $V^+_{12} \cup V^+_{123}$ possibly empty) and $\{v_3, v_4\} \cup V^+_{34} \cup V^+_{234}$ (with $V^+_{34} \cup V^+_{234}$ possibly empty), and tips v_1 and v_4. Since G is not the result of a nontrivial raft expansion, we conclude that $V^+_{12} = V^+_{123} = V^+_{34} = V^+_{234} = \emptyset$. This implies that that G is isomorphic to the four-vertex path. This is a contradiction that completes the proof of the theorem. ■

We have proved Theorem 2, which provides a decomposition result for (bull, fork, C_4)-free graphs. However, this theorem is not a *composition* theorem since certain graphs not in the class can also be produced by means of the operations described in the theorem—while the class of (bull, fork, C_4)-free graphs is closed under the disjoint union, vertex duplication, and addition of dominating vertices, it is not closed under raft expansions and semiraft expansions:

- If G is a (bull, fork, C_4)-free graph containing an edge $e = xy \in E(G)$ such that $G - e \in$ {bull, fork, C_4} then applying a semiraft expansion with respect to x, y will produce in a graph containing an induced copy of $G - e$.
- Similarly, if G is a (bull, fork, C_4)-free graph containing two nonadjacent vertices x, y such that $G + xy \in$ {bull, fork, C_4} then applying a raft expansion with respect to x, y will produce a graph containing an induced copy of $G + xy$.

Nevertheless, if only *safe* raft expansions and semiraft expansions are allowed—safe in the sense that they do not produce a bull, a fork, or a C_4 when applied to a (bull, fork, C_4)-free graph—then Theorem 2 can be turned into a stronger theorem (Theorem 3 below), in which the decomposition can also be reversed in the sense that a graph is in the class if and only if it can be constructed by gluing basic graphs along the decompositions prescribed. A *long cycle* is a cycle of order 5 or more.

Theorem 3. *Let G be a graph. Then, G is (bull, fork, C_4)-free if and only if G can be built from paths and long cycles by applying a sequence of the following operations:*

- *disjoint union of two graphs,*
- *vertex duplication,*
- *addition of a dominating vertex,*
- *safe raft expansion,*
- *safe semiraft expansion.*

4. CHARACTERIZATIONS OF HEREDITARY ED GRAPHS

In this section, we prove several characterizations of hereditary ED graphs. First, we derive from Theorem 2 the characterization of hereditary ED graphs in terms of forbidden induced subgraphs. We start with a preliminary lemma.

Lemma 4. *The set of ED graphs is closed under each of the following operations:*

- *disjoint union of two graphs,*
- *vertex duplication,*
- *addition of a dominating vertex,*
- *raft expansion,*
- *semiraft expansion.*

Proof. Let G and H be two vertex-disjoint ED graphs, with respective efficient dominating sets D_G and D_H.

Then clearly $D_G \cup D_H$ is an efficient dominating set in the disjoint union of G and H. Also, it is easy to see that D_G is an efficient dominating set in the graph obtained from G by duplicating a vertex.

Adding a dominating vertex v^* to any graph produces a graph with an efficient dominating set (namely, the set $\{v^*\}$).

Suppose that G' is the graph resulting from G by a raft expansion with respect to two nonadjacent vertices x, y, where x_0 and y_0 are the tips of the resulting raft. Then, an efficient dominating set D' in G' can be obtained by setting

$$D' = \begin{cases} D_G, & \text{if } D_G \cap \{x, y\} = \emptyset; \\ (D_G \setminus \{x\}) \cup \{x_0\}, & \text{if } D_G \cap \{x, y\} = \{x\}; \\ (D_G \setminus \{y\}) \cup \{y_0\}, & \text{if } D_G \cap \{x, y\} = \{y\}; \\ (D_G \setminus \{x, y\}) \cup \{x_0, y_0\}, & \text{if } D_G \cap \{x, y\} = \{x, y\}. \end{cases}$$

Finally, suppose that G' is the graph resulting from G by a semiraft expansion with respect to two adjacent vertices x, y, where x_1 and y_n are the tips of the resulting semiraft. Then, an efficient dominating set D' in G' can be obtained by setting

$$D' = \begin{cases} D, & \text{if } D \cap \{x, y\} = \emptyset; \\ (D \setminus \{x\}) \cup \{x_1\}, & \text{if } D \cap \{x, y\} = \{x\}; \\ (D \setminus \{y\}) \cup \{y_n\}, & \text{if } D \cap \{x, y\} = \{y\}. \end{cases}$$

■

Theorem 5. *The class of hereditary ED graphs equals the class of (bull, fork, C_{3k+1}, C_{3k+2})-free graphs.*

Proof. It suffices to show that every (bull, fork, C_{3k+1}, C_{3k+2})-free graph is ED; the other direction has already been observed in the Introduction.

Let G be a (bull, fork, C_{3k+1}, C_{3k+2})-free graph. By Theorem 2, G can be built from paths and cycles by applying a sequence of the following operations: disjoint union of two graphs, vertex duplication, addition of a dominating vertex, raft expansion, semiraft expansion. As shown in Lemma 4, each of these operations preserves the property of being ED. Therefore, by induction on $|V(G)|$, it is enough to verify that the basic graphs are efficiently dominatable. But this is indeed the case since if a (bull, fork, C_{3k+1}, C_{3k+2})-free graph G is a path or a cycle, then G is ED. Recall that all paths are ED, and a cycle is ED if and only if its order is divisible by 3. ■

In particular, Theorem 5 implies that the class of hereditary ED graphs properly extends the class of (P_4, C_4)-free graphs (also known as trivially perfect graphs [31] or quasi-threshold graphs [51]), and their subclass threshold graphs [14].

Similarly as in the case of (bull, fork, C_4)-free graphs, it is not hard to see that the class of (bull, fork, C_{3k+1}, C_{3k+2})-free graphs is closed under the disjoint union, vertex duplication, and addition of dominating vertices, however it is not closed under raft expansions and semiraft expansions. By forbidding such "bad" raft expansions and semiraft expansions, we obtain, for hereditary ED graphs, a similar composition result as that given by Theorem 3 for (bull, fork, C_4)-free graphs. We say that a raft expansion or a semiraft expansion is *very safe* if it does not produce an induced bull, fork, or a cycle whose order is not divisible by 3. In other words, when a very safe expansion is applied to a (bull, fork, C_{3k+1}, C_{3k+2})-free graph, it results in a (bull, fork, C_{3k+1}, C_{3k+2})-free graph.

Summarizing this discussion and the result of Theorem 5, we collect the different characterizations of hereditary ED graphs in the following theorem:

Theorem 6. *For every graph G, the following properties are equivalent:*

(i) *G is hereditary ED.*
(ii) *G is (bull, fork, C_{3k+1}, C_{3k+2})-free.*
(iii) *G can be built from paths and cycles whose order is a multiple of* 3 *by applying a sequence of the following operations:*
- *disjoint union of two graphs,*
- *vertex duplication,*
- *addition of a dominating vertex,*
- *very safe raft expansion,*
- *very safe semiraft expansion.*

5. ALGORITHMIC ASPECTS

Based on the above decomposition theorem, we now develop a robust algorithm for finding an efficient dominating set in a given hereditary ED graph. A pseudocode for the algorithm is given in Algorithm 1 below. The algorithm is robust [49] in the sense that it either finds an efficient dominating set of the input graph, or it outputs a certificate for the fact that the graph is not hereditary ED (that is, an induced fork, a bull, or a cycle whose order is not divisible by 3). However, it should be mentioned that the algorithm may also find an efficient dominating set in an input graph that is not HED (for instance, in the graph obtained from the 5-cycle C_5 by adding to it a dominating vertex).

For the ease of presentation, we will assume in the algorithm below that in each recursive call of the algorithm, an ED set of the input graph (for that recursive call) was found. This is without loss of generality since if in some recursive call, a certificate is found for the fact that the input graph (for that recursive call) is not hereditary ED, this also shows that the input graph G is not hereditary ED, and there is no need to proceed with the computation.

Algorithm 1. A robust algorithm for finding an efficient dominating set in a hereditary efficiently dominatable graph

Input: A graph $G = (V, E)$.
Output: An efficient dominating set D in G, or a proof that G is not hereditary efficiently dominatable (i.e., a subset $S \subseteq V$ inducing a bull, a fork or a cycle whose order is not divisible by 3).
1: **if** G contains an induced bull, fork, or C_4 **then**
2: **return** any subset $S \subseteq V$ inducing a bull, a fork, or a C_4 (a certificate that $G \notin \mathsf{HED}$)
3: **else if** G is disconnected **then**
4: solve the problem recursively on the connected components
5: **return** the union of the obtained ED sets
6: **else if** G contains a dominating vertex v **then**
7: **return** an efficient dominating set $\{v\}$
8: **else if** G contains a duplicated vertex v **then**
9: solve the problem recursively on $G - v$
10: **return** the obtained ED set
11: **else if** G contains a nontrivial raft or semiraft with parts X and Y and tips $x \in X$ and $y \in Y$ **then**
12: solve the problem recursively on $G - ((X \setminus \{x\}) \cup (Y \setminus \{y\}))$
13: **return** the obtained ED set
14: **else if** G is a path or a cycle whose order is divisible by 3 **then**
15: **return** an efficient dominating set in G
16: **else**
17: **return** $S = V$ (a certificate that $G \notin \mathsf{HED}$)
18: **end if**

The correctness of the algorithm follows directly from Lemma 4 and Theorems 2 and 5. In particular, the algorithm returns $S = V$ in line 17 if and only if G is a cycle whose order is not divisible by 3. To justify the polynomial running time of the algorithm, observe that each of the conditions of **if** statements in lines 1, 3, 6, 8, and 14 can be verified in polynomial time. The only nontrivial part is to show that verifying the condition of **if** statement in line 11 is also a polynomially solvable task.

Proposition 7. *It can be verified in polynomial time whether a given connected graph G without dominating or duplicated vertices contains a nontrivial raft or semiraft. A nontrivial raft or semiraft, if one exists, can be found in polynomial time.*

Proof. First, we describe the idea for rafts. The algorithm is based on the observation that for every induced P_4 in G, with vertices a, b, c, d in this order, there is at most one raft in G with parts X and Y such that $a \in X$ and $d \in Y$ are the tips of the raft, $b \in X$ dominates $Y \setminus \{c\}$, and $c \in Y$ dominates $X \setminus \{b\}$. Indeed, if $G[X \cup Y]$ is a raft in G with the above properties, then we must have $X \setminus \{a, b\} = N_G(a) \cap N_G(b) \cap N_G(c) \setminus N_G(d)$. (The inclusion $\subseteq$ follows from the definition of a raft, and the inclusion $\supseteq$ from the definition of the raft expansion.) Similarly, we have $Y \setminus \{c, d\} = N_G(b) \cap N_G(c) \cap N_G(d) \setminus N_G(a)$. Therefore, the sets X and Y are uniquely determined by the neighborhoods of a, b, c, and d, and we only need to verify the remaining necessary conditions for $X \cup Y$ to induce a raft in G:

- X and Y are cliques of the same size,
- every vertex $z \in V(G) \setminus (X \cup Y)$ is adjacent either to all or to no vertices in X, and either to all or to no vertices in Y,
- vertices in $X \setminus \{a\}$ can be linearly ordered with respect to inclusion of their neighborhoods in $Y \setminus \{d\}$.

The set $X \cup Y$ induces a raft in G if and only if all the above conditions are met. Each of these conditions can be verified in polynomial time. Therefore, by exhaustively checking all induced P_4s in G, we can determine in polynomial time whether G is the result of a nontrivial raft expansion, and a nontrivial raft can be found if one exists.

Similar ideas can be used to determine G is the result of a nontrivial semiraft expansion. In few words, observe that for every induced diamond (that is, K_4 minus an edge) in G, with vertices a, b, c, d and edges ab, ac, ad, bd, cd, there is at most one raft in G with parts X and Y such that $a \in X$ and $d \in Y$ are the tips of the semiraft, $b \in X$ is adjacent to no vertex in $Y \setminus \{d\}$, and $c \in Y$ is adjacent to no vertex in $X \setminus \{a\}$. Indeed, if $G[X \cup Y]$ is a semiraft in G with the above properties, then we must have $X \setminus \{a, b\} = N_G(a) \cap N_G(b) \cap N_G(d) \setminus N_G(c)$. Similarly, we have $Y \setminus \{c, d\} = N_G(c) \cap N_G(d) \cap N_G(a) \setminus N_G(b)$. The rest of the algorithm is the same as the one given above for rafts, except that induced P_4s are replaced with induced diamonds. ■

This discussion is summarized in the following theorem.

Theorem 8. *There exists a polynomial time algorithm that, given a graph G, either finds an efficient dominating set in G, or finds in G an induced copy of a graph from the set* $\{bull, fork\} \cup \bigcup_{k \geq 1}\{C_{3k+1}, C_{3k+2}\}$.

In what follows, we will generalize the algorithmic result given by Theorem 8, using a completely different approach. The generalization will be twofold: we will solve a more general problem in a larger class of graphs.

The problem of determining whether a given graph is ED is a special case of the problem of finding the maximum number of vertices in a graph that can be efficiently dominated. Bange et al. [1] introduced the *efficient domination number* of a graph, denoted by $F(G)$, as the maximum number of vertices that can be dominated by a set D that dominates each vertex at most once. Clearly, a graph G is ED if and only if $F(G) = |V(G)|$. The *efficient domination problem* is the problem of computing, in a given graph, a set D that dominates $F(G)$ vertices and dominates each vertex at most once.

We now show that the efficient domination problem is a special case of the maximum weight independent set problem in the square of the graph.

The *square* of a graph $G = (V, E)$ is the graph $G^2 = (V, E^2)$ such that $uv \in E^2$ if and only if either $uv \in E$ or u and v have a common neighbor in G. As we show in the proposition below, the efficient domination number of G equals the maximum weight of an independent set in G^2, where each vertex is weighted according to the cardinality of its neighborhood. Recall that an independent set in a graph G is a subset of vertices no two of which are adjacent. If G is a vertex-weighted graph and $I \subseteq V(G)$, then $w(I)$ is the weight of the set I, i.e., the sum of weights of its vertices, and $\alpha_w(G)$ is the maximum weight of an independent set in G.

Proposition 9. *Let G be a graph. Define $w(x) := |N[x]|$ for all $x \in V(G)$. Then, a set $D \subseteq V(G)$ dominates $F(G)$ vertices, each vertex at most once, if and only if D is an*

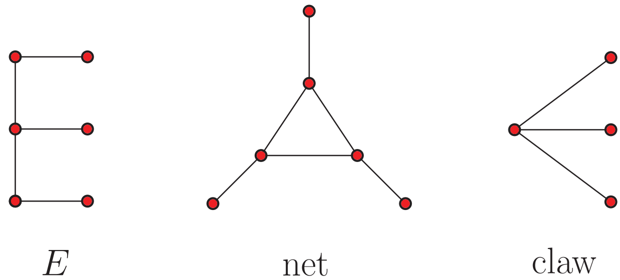

FIGURE 5. The graphs E, the net, and the claw.

independent set of maximum weight in G^2. In particular, $F(G) = \alpha_w(G^2)$, and G is ED if and only if $\alpha_w(G^2) = |V(G)|$.

Proof. Suppose that $D \subseteq V(G)$ dominates $F(G)$ vertices, each vertex at most once. Then, D is independent in G and no two vertices in D have a common neighbor in G. Thus, D is an independent set in G^2 of total weight $w(D) = \sum_{x\in D} |N(x)| = F(G)$. Consequently, $\alpha_w(G^2) \geq F(G)$.

Now, let $D \subseteq V(G)$ be a maximum weight independent set in G^2 (with respect to w). Then, D is an independent set in G and no two vertices in D have a common neighbor. Thus, each vertex of G is dominated by at most one vertex from D, and D dominates exactly k vertices, where $k = \sum_{x\in D} |N(x)| = w(D) = \alpha_w(G^2)$. Therefore, $F(G) \geq \alpha_w(G^2)$, which implies $F(G) = \alpha_w(G^2)$.

Combining the two parts implies that if a set $D \subseteq V(G)$ dominates $F(G)$ vertices, each vertex at most once, then it is an independent set of maximum weight in G^2, and conversely, if D is an independent set of maximum weight in G^2, then D dominates $w(D) = \alpha_w(G^2) = F(G)$ vertices, each vertex at most once. ■

We now identify a class of graphs the squares of which are claw-free. See Figure 5 for definitions of the graphs E, the net, and claw.

Proposition 10. *Let G be an (E, net)-free graph. Then G^2 is claw-free.*

Proof. Suppose for a contradiction that G is an (E, net)-free graph such that G^2 contains an induced claw K on the vertex set $\{a, b, c, d\}$, where a is the vertex of degree 3 in K.

We analyze the possible distances in G between vertices of K. Clearly, a is at distance either 1 or 2 from each of $\{b, c, d\}$, however, since the set $\{b, c, d\}$ is independent in G^2, vertex a can be adjacent in G to at most one of $\{b, c, d\}$.

Suppose first that a is adjacent in G to precisely one of $\{b, c, d\}$, say to b. Denote by e a common neighbor of a and c in G and by f a common neighbor of a and d in G. Since c and d are not adjacent in G^2, $e \neq f$, $ed \notin E(G)$, and $fc \notin E(G)$. Since $bc \notin E(G^2)$, we conclude that $be \notin E(G)$ and similarly $bf \notin E(G)$. Now, depending on the presence or the absence of the edge ef in $E(G)$, we see that G contains either an induced net or an induced E, a contradiction.

Suppose now that a is at distance two in G from each of $\{b, c, d\}$. Let us denote by e, f, g respective common neighbors in G of a and b, a and c, and a and d. Then e, f, g are all distinct, e is nonadjacent to c and d, f is nonadjacent to b and d, and e is nonadjacent to b and c. Let k denote the number of edges in G within the set $\{e, f, g\}$. If $k \in \{0, 2\}$, then G contains an induced E, and if $k \in \{1, 3\}$, then it contains an induced net. In either case, we reach a contradiction. This completes the proof of the proposition. ■

Since the maximum weight independent set problem is solvable in polynomial time for claw-free graphs [29, 43–47], we obtain the following result.

Theorem 11. *There is a polynomial time algorithm for the efficient domination problem in (E, net)-free graphs.*

In view of Theorem 5, it is important to notice that the class of (E, net)-free graphs contains the class of (bull, fork)-free graphs, and with it also the class of hereditary ED graphs. This way, we obtain an alternative proof of the polynomial time solvability of the problem of finding an efficient dominating set in a given hereditary ED graph.

6. CONCLUDING REMARKS

In this article, we have given several characterizations of the class of hereditary ED graphs, that is, graphs every induced subgraph of which contains an efficient dominating set. These graphs are characterized by infinitely many forbidden induced subgraphs: the bull, the fork, and all cycles whose order is not divisible by 3. Every hereditary ED graph can be built from paths and cycles the length of which is a multiple of 3 by means of five types of operations: disjoint union of two graphs, vertex duplication, addition of a dominating vertex, raft expansion, and semiraft expansion.

A natural related question would be to characterize the class HETD of *hereditary efficiently total dominatable graphs*, defined analogously as the class of hereditary ED graphs: a graph G is hereditary efficiently total dominatable if and only if every induced subgraph of G contains an efficient total dominating set. (An *efficient total dominating set* in a graph G is a set $D \subseteq V(G)$ such that every vertex of the graph has precisely one neighbor in D.) It is not hard to see that, contrary to the class HED, the class HETD is characterized by only finitely many forbidden induced subgraphs, namely K_3, P_5, and C_5. In other words, the hereditary efficiently total dominatable graphs are precisely the P_5-free bipartite graphs. Every connected component of such a graph is a $2K_2$-free bipartite graph, and hence can be obtained from the complement of a semiraft by substituting some vertices with independent sets.

As a final remark, we observe that hereditary ED graphs can be recognized in linear time by combining the following results:

1. We can express the defining property of hereditary ED graphs in monadic second-order logic (MSOL) with quantifiers over vertices and vertex subsets, as follows:

$$(\forall X \subseteq V)(\exists D \subseteq X)(\mathit{Independent}(D) \wedge \mathit{Dominating}(D, X) \\ \wedge \mathit{EfficientDominating}(D, X)),$$

   where

$$\mathit{Independent}(D) := (\forall x \in D)(\forall y \in D)(\neg E(x, y)),$$

$$\mathit{Dominating}(D, X) := (\forall x \in X \setminus D)(\exists y \in D)(E(x, y)),$$

$$\mathit{EfficientDominating}(D, X) := (\forall x \in X \setminus D)(\forall y \in D)(\forall z \in D)(E(x, y) \wedge E(x, z) \Rightarrow y = z) .$$

2. The class of hereditary ED graphs forms a subclass of (bull, fork, house)-free graphs. (A *house* is the graph obtained from a triangle and a 4-cycle by gluing them along an edge.) Therefore, since (bull, fork, house)-free graphs are of bounded clique-width, and a k-expression of a given (bull, fork, house)-free graph can be obtained in linear time [5, 6], the same is true for hereditary ED graphs.
3. For every k, every (decision or optimization) problem expressed in MSOL with quantifiers over vertex sets is solvable in linear time, given a k-expression of the input graph [26].

We leave it as an open problem to develop a polynomial time recognition algorithm for hereditary ED graphs that does not rely on the MSOL framework.

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